

A SEPARABLE MANIFOLD FAILING TO HAVE THE HOMOTOPY TYPE OF A CW-COMPLEX

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ABSTRACT. We show that the Prüfer surface, which is a separable non-metrizable 2-manifold, has not the homotopy type of a CW-complex. This will follow easily from J. H. C. Whitehead's result: if one has a good approximation of an arbitrary space by a CW-complex, which fails to be a homotopy equivalence, then the given space is not homotopy equivalent to a CW-complex.

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1 Introduction

Our aim is to prove the following:

Theorem 1.1. *The Prüfer surface¹ (which is an example of separable² non-metrizable manifold³) has not the homotopy type of a CW-complex.*

This might sound like a dissonance in view of Milnor's Corollary 1 [4], which says that *every separable manifold has the homotopy type of a (countable) CW-complex*.

However the proof of Corollary 1 uses metrizability in a crucial way. Remember that it works as follows: by results of J. H. C. Whitehead [10] it is enough to prove that our space is dominated by a CW-complex. The first step is Hanner's theorem [3]: a space that is locally an ANR is an ANR. Then following Kuratowski, the space is embedded via $x \mapsto d(x, \cdot)$ making use of a (bounded) metric d into a Banach space as a closed subset of its convex hull C (Wojdyslawski [9]). Since it is an ANR, there is an open neighborhood U in C retracting to our space. By transitivity of domination, it is enough to prove that U is dominated $f : P \rightarrow U$ by a polyhedron P , which is constructed as the nerve of a suitable cover. The "submission" map $g : U \rightarrow P$ is constructed as the barycentric map attached to a partition of unity (paracompactness is needed, but follows from metrizability). Lastly the homotopy $fg \simeq 1_U$ comes from local convexity considerations. For a detailed exposition see [6]. (All this, being an elaboration of the basic idea: embed the given space in an Euclidean space and triangulate an open tubular neighborhood of it.)

From this context it is quite clear that all manifolds in Milnor's paper are implicitly assumed to be metrizable. In particular Corollary 1 does not apply to the manifold constructed (under CH=continuum hypothesis) by Rudin-Zenor [8], which is an example of hereditarily separable⁴ non-metrizable manifold. The question of the contractibility of the Rudin-Zenor manifold then appears as an interesting problem.

2 The idea of the proof

Let us first give a loose description of the Prüfer surface P . We may think of P as the (Euclidean) plane from which an (horizontal) line has been suppressed, and then for each point of the line a small bridge is introduced in order to connect the upper half-plane \mathcal{H} to the lower half-plane \mathcal{H}^σ . (The formal construction of P will be recalled in § 3.) For our argument, the only information which will be needed on P is its separability. The idea is to consider the natural map f from the graph K consisting of two vertices and a continuum \mathfrak{c} (=cardinality of \mathbb{R}) of edges linking them,

¹Actually, the surface we consider is not exactly the original Prüfer surface, but rather Calabi-Rosenlicht's slight modification of it.

²A space is *separable* if it has a countable dense subset.

³Here this means a Hausdorff topological space which is locally Euclidean.

⁴Each subspace is separable.

to P given by going from a point in \mathcal{H} to a point in \mathcal{H}^σ crossing through the continuum of bridges at our disposal (see Figure 1).

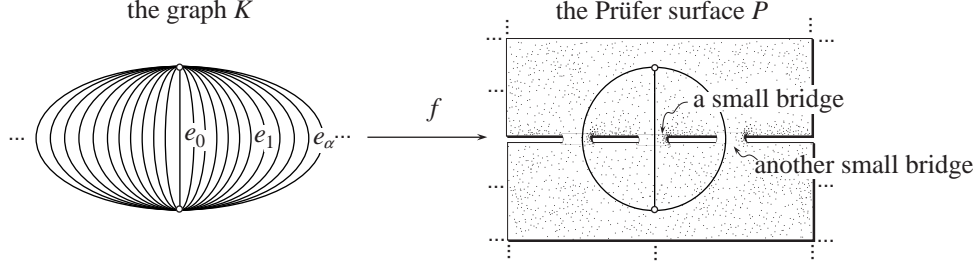


Figure 1. Constructing a weak homotopy equivalence

The fundamental group of P is easily identified, via van Kampen's theorem, as a free group on a continuum \mathfrak{c} of generators. Further, it is not hard to check that the higher homotopy groups of P vanish, i.e. $\pi_i(P) = 0$ for all $i \geq 2$. (Details will be given in § 4.) It follows that the map $f : K \rightarrow P$ is a weak homotopy equivalence⁵, which turns out *not* to be (as we will soon explain) a strong homotopy equivalence. Then Whitehead's Theorem 1 [10] implies that P has not the homotopy type of a CW-complex. To explain why f is not a homotopy equivalence, we first observe that if there were a homotopy inverse $g : P \rightarrow K$, then g has to be onto. [This, because if g misses a point $p \in K$, then g factors through $K - p$, and so by functoriality the inclusion $K - p \hookrightarrow K$ has to induce an epimorphism on the H_1 (first homology group). But our graph K is easily verified to be such that for any point $p \in K$ the inclusion $K - p \hookrightarrow K$ fails to induce an epimorphism on the H_1]. But then pulling-back the uncountable collection of open 1-cells $(e_\alpha)_{\alpha \in \mathbb{R}}$ of K , we get in P an uncountable family $(g^{-1}(e_\alpha))_{\alpha \in \mathbb{R}}$ of pairwise disjoint open sets. This is a contradiction, since P is separable. \square

3 Construction of the Prüfer surface P

The following construction is due to Prüfer, first described in print by Radó [7]. (As it is well-known, this was also the “first” example of a non-triangulable surface, and played an important role in clarifying foundational aspects of Riemann surfaces theory.) We use \mathbb{C} as model for the Euclidean plane. The idea is to consider the set P_0 formed by the (open) upper half-plane $\mathcal{H} = \{z : \text{Im}(z) > 0\}$ together with the set of all rays emanating from point of \mathbb{R} and pointing out in the upper half-plane. Then we topologize P_0 with the usual topology for \mathcal{H} , and by taking as neighborhoods of a point r which is a ray (say emanating from $x \in \mathbb{R}$) an (open) sector of rays deviating by at most ε radian from r , together with the points of \mathcal{H} between the two rays and at (Euclidean) distance smaller than ε from x (see Figure 2).

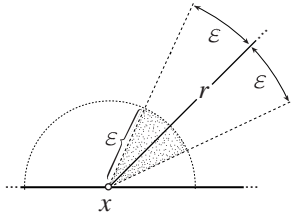


Figure 2. A neighborhood of a ray

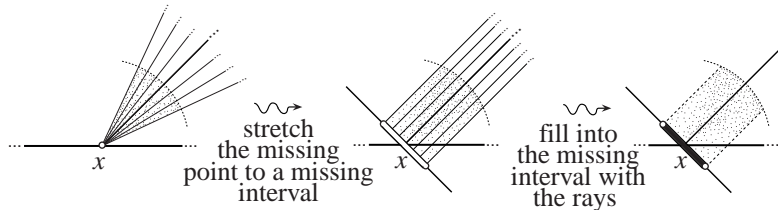


Figure 3. Proving that P_0 is a manifold with boundary

⁵i.e. induces isomorphisms on all homotopy groups π_i .

The space P_0 is a surface-with-boundary, as it is easily seen by following the pictures from Figure 3. Observe that P_0 has a continuum \mathfrak{c} of boundary components each homeomorphic to the real line \mathbb{R} .

Now given a manifold-with-boundary W , there are two obvious ways to obtain a manifold M (without boundary): a first method is by *collaring*, set $M = W \cup_{\text{id}_{\partial W}} (\partial W \times [0, 1])$ (glue W with the cylinder having the boundary ∂W as basis, along their boundaries), and a second option is by *doubling*, $M = W \cup_{\text{id}_{\partial W}} W$ (two copies of W are glued along their boundaries).

For $W = P_0$, the process of collaring leads to the “original” Prüfer surface (the one described in [7]). In this case there is in M an uncountable family of pairwise disjoint open sets. This implies that M is non-separable. (And so it did not really interest us: remember our purpose is to point out that the metrizability hypothesis in Milnor’s Corollary 1 is essential.)

The second option leads to the surface P we are interested in since it is separable. We call it also the Prüfer surface (even though it seems to appear explicitly only in the paper by Calabi-Rosenlicht [2]).

Proposition 3.1. *The Prüfer surface P obtained by the process of doubling, is a connected (Hausdorff) 2-manifold which is separable, but contains an uncountable discrete subspace (and therefore is non-metrizable).*

Proof. Observe that the rational points $\mathbb{Q} + i\mathbb{Q}_{>0}$ give a countable dense subset of P_0 , and so P is clearly separable. Further notice that the set of all rays $(r_x)_{x \in \mathbb{R}}$, say orthogonal to \mathbb{R} gives an uncountable discrete subset of P , since an open neighborhood of such a ray cuts out only this single one from the whole family. It follows that P is not hereditarily separable, and so not second countable, and therefore non-metrizable. (As it is well-known metrizability and second countability are equivalent for (connected) manifolds. Actually in our situation since P is separable, the non-metrizability of P can also be deduced from the elementary fact that metrizable plus separable imply second countable.) \square

At this stage one could already observe the following:

Corollary 3.2. *The Prüfer surface P (and more generally any non-metrizable manifold) is not homeomorphic to a CW-complex.*

Proof. This follows from the fact proved by Miyazaki [5] that CW-complexes are always paracompact, and the equivalence between the concepts of paracompactness and metrizability, when spaces are restricted to be manifolds. \square

4 Homotopy groups of the Prüfer surface

We now investigate the homotopy groups of P . First, the fundamental group π_1 :

Proposition 4.1. *$\pi_1(P)$ is a free group on a continuum \mathfrak{c} of generators.*

Proof. (Following [1]). For all $x \in \mathbb{R}$, let U_x be the open neighborhood depicted in Figure 2 with r chosen orthogonal to \mathbb{R} and $\varepsilon = \frac{\pi}{2}$. Let then B_x be U_x taken together with its symmetrical copy U_x^σ , so $B_x = U_x \cup U_x^\sigma$ is an open set of P (we can think of it as a “bridge” linking the upper to the lower half-plane). For all $x \in \mathbb{R} - \{0\} = \mathbb{R}^*$, put $O_x = \mathcal{H} \cup \mathcal{H}^\sigma \cup B_0 \cup B_x$. The collection $(O_x)_{x \in \mathbb{R}^*}$ form an open cover of P , which satisfies van Kampen’s theorem hypothesis, since $O_x \cap O_y$ is arcwise-connected. Furthermore $\cap_{x \in \mathbb{R}^*} O_x = \mathcal{H} \cup \mathcal{H}^\sigma \cup B_0$, this being homeomorphic to $\mathbb{C} - \mathbb{R}$ union an open interval from \mathbb{R} (by the same kind of argument as the one cartooned in Figure 3), and so in particular simply connected. Moreover each member O_x of this cover is homeomorphic to $\mathbb{C} - \mathbb{R}$ union two disjoint (real) intervals, and so has the homotopy type of the circle S^1 . The result follows by van Kampen’s theorem. \square

This proof gives not only the abstract algebraic structure of the group $\pi_1(P)$, but also a concrete geometric description for generators. They can be chosen as the loops going first down from $i = \sqrt{-1}$ to $-i$ straightforwardly, and then going up by travelling along an elliptic arc crossing the real axis through some non-zero real number x and turning back again to i (see Figure 1). (One further has to agree that all times we cross the missing real axis at some abscissae $x \in \mathbb{R}$, we cross it through the ray orthogonal to \mathbb{R} emanating from x .) Tautologically, it follows that the map $f : K \rightarrow P$, being essentially defined in the same way, induces an isomorphism on the fundamental groups.

Next the higher homotopy groups are simply given by:

Proposition 4.2. $\pi_i(P) = 0$ for all $i \geq 2$.

Proof. We consider the open cover \mathcal{U} of P given by \mathcal{H} , \mathcal{H}^σ and for all $x \in \mathbb{R}$ the bridge $B_x = U_x \cup U_x^\sigma$ (previously defined). Let now $\gamma \in \pi_i(P)$, and choose $c : S^i \rightarrow P$ a representing map. Pulling back the cover \mathcal{U} by the map c , we see by compactness of the sphere S^i that there is a finite subset \mathcal{F} of \mathcal{U} such that the image $c(S^i)$ is contained in $\cup_{U \in \mathcal{F}} U =: V$. But it is clear (again by Figure 3) that V is homeomorphic to $\mathbb{C} - \mathbb{R}$ union a finite number of open intervals (if not something even simpler like only \mathcal{H} or \mathcal{H} union a finite number of bridges). In any case the space V has the homotopy type of a wedge of circles (maybe an empty one), and so has vanishing π_i for $i \geq 2$. We are done. \square

The proof of Theorem 1.1 is then complete. \square

5 The case of non-Hausdorff manifolds

We conclude by making simple observations concerning complications arising in the relation between manifolds and CW-complexes, if we drop Hausdorff from the definition of a manifold. Then already one of the simplest example of “manifold”, the so called *line with two origins* (obtained from two copies of \mathbb{R} by identifying corresponding points outside the origin, see Figure 4) fails to have the homotopy type of a CW-complex (and this in spite of the fact that it is well-behaved from the point of view of second countability). Actually we even have a worse situation:

Proposition 5.1. *The line with two origins R has not the homotopy type of any Hausdorff topological space.*

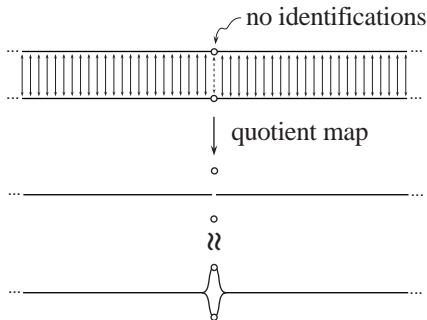


Figure 4. The line with two origins as a quotient space

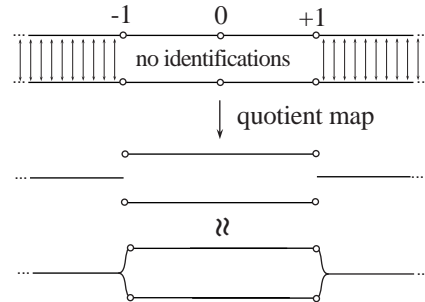


Figure 5. Another non-Hausdorff manifold

Proof. We need two preliminary remarks.

- First remember that there is a general Hausdorffization process applicable to any space X , which leads to an Hausdorff space X_{Haus} with a map $X \rightarrow X_{\text{Haus}}$. (This is by moding out the given space by the equivalence relation generated by the inseparability relation.) It has the property that any (continuous) map from X to an Hausdorff space H factors through X_{Haus} .

- Second by Mayer-Vietoris it is easy to check that $H_1(R) \cong \mathbb{Z}$.

We are now in position to prove 5.1. Assume there is a homotopy equivalence $f : R \rightarrow H$ between R and some Hausdorff space H . Then f factors through R_{Haus} , which is nothing else than the usual real line \mathbb{R} . But this being contractible, it follows by functoriality that the morphism $H_1(f)$ is zero, in contradiction to the non-vanishing of $H_1(R)$. \square

Furthermore it is not difficult to compute the homotopy groups of R (for example by looking at R as the leaf space of $\mathbb{C} - 0$ foliated by vertical lines, and then applying the exact homotopy sequence of a fibration). We conclude that R is an Eilenberg-Mac Lane space of type $K(\mathbb{Z}, 1)$ not homotopy equivalent to the circle S^1 .

Finally, a variant of the line with two origins (see Figure 5) leads to a non-Hausdorff manifold which is easily seen to be homotopy equivalent to the circle S^1 . So, it is not non-Hausdorffness as a rule, that leads us outside the class \mathcal{W} of spaces having the homotopy type of a CW-complex, but much more the strange geometric behavior of “extremely narrow bifurcations” presented by some non-Hausdorff manifolds, which appears as something alien to the combinatorial nature of CW-complexes.

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